COMBINATORICA
Bolyai Society – Springer-Verlag

COMBINATORICA 18 (1) (1998) 133–137

ON SMALL BLOCKING SETS

POMPEO POLITO and OLGA POLVERINO

Received August 5, 1997

We construct small minimal blocking sets of a desarguesian plane, which are not of Rédei type.

1. Introduction

A blocking set $\mathcal{B}$ in a projective plane is a set of points which intersects every line. If $\mathcal{B}$ contains a line, it is called *trivial* and if no proper subset of it is a blocking set, it is called *minimal*.

Let $\mathcal{B}$ be a non-trivial blocking set in a projective plane of order $q=p^n$. It is easy to see that $|\mathcal{B}\cap L|\leq|\mathcal{B}|-q$ for every line L. If some line L contains exactly $|\mathcal{B}|-q$ points of $\mathcal{B}$, the blocking set is called of *Rédei type* and the line L is called a *Rédei line*. We will call *small* a blocking set of size less than $\frac{3(q+1)}{2}$. In [8] T. Szőnyi has proved the following result for the small minimal blocking sets $\mathcal{B}$ of the projective plane $PG(2,q)$, with $q=p^n$: there exists an integer e $(1\leq e\leq n/2)$ such that the size of $\mathcal{B}$ must lie in an interval depending on p^e and if $p^e\neq 4,8$ for every line L we have $|\mathcal{B}\cap L|\equiv 1 \bmod p^e$.

If $\mathcal{B}$ is a small minimal blocking set of Rédei type, it has a very special structure. If $p>3$ and L is a line such that $|\mathcal{B}\cap L|=|\mathcal{B}|-q$, the subset $U=\mathcal{B}\setminus L$ of the affine plane $PG(2,q)\setminus L$, is a $GF(p^e)$-linear subspace with $q=p^n$, $e|n$, $0<e\leq n/2$, and every line intersects $\mathcal{B}$ in 1 modulo p^e points. For $p=2$ or $p=3$ a slightly weaker result holds, [1].

The known examples of small blocking sets are of Rédei type and, in the projective planes $PG(2,p^2)$ and $PG(2,p^3)$, $p\geq 7$, all small minimal blocking sets are of Rédei type, (see [8] and [6], resp.). In [3] A. Blokhuis conjectured that all small minimal blocking sets are of Rédei type.

Mathematics Subject Classification (1991): 05B25, 51E21

In this paper we disprove this conjecture constructing examples of small minimal blocking sets which are not of Rédei type.

2. Linear blocking sets

A $(t-1)$-spread $\mathcal{S}$ of $PG(rt-1,q)$ is a set of $(t-1)$-dimensional subspaces such that each point of $PG(rt-1,q)$ is contained in exactly one element of $\mathcal{S}$. A $(t-1)$-spread $\mathcal{S}$ is called *normal* if for every pair (X,Y) of distinct elements of $\mathcal{S}$, the set

$$\mathcal{S}_T = \{Z \in \mathcal{S} \,:\, Z \cap T \neq \emptyset\}$$

is a $(t-1)$-spread of T, where $T = \langle X, Y\rangle$.

Let $\mathcal{S}$ be a normal $(t-1)$-spread of the projective space $PG(3t-1,q)$ and, denoted by $\mathcal{P}(\mathcal{S})$ the family of all subsets of $\mathcal{S}$, let $\mathcal{L}$ be the following subset of $\mathcal{P}(\mathcal{S})$:

$$\mathcal{L} = \{\mathcal{S}_T \,:\, T = \langle X, Y\rangle\,, \quad X, Y \in \mathcal{S} \;\text{ and }\; X \neq Y\}.$$

Then $\Pi = (\mathcal{S},\mathcal{L})$ is a desarguesian plane of order q^t (see [5]).

If L is a t-dimensional subspace of $PG(3t-1,q)$, we denote by $\mathcal{B}_L$ the subset of $\mathcal{S}$ defined by:

$$\mathcal{B}_L = \{X \in \mathcal{S} \,:\, X \cap L \neq \emptyset\}.$$

It is easy to see that $\mathcal{B}_L$ is a blocking set which we will call *linear* ([6]).

Lemma 1. *$\mathcal{B}_L$ is a minimal blocking set and $\mathcal{B}_L$ coincides with a line $\mathcal{S}_T$ if and only if $L \subset T$. Moreover the size of $\mathcal{B}_L$ is at most $q^t + q^{t-1} + \cdots + q + 1$.*

Proof. If $\mathcal{B}_L$ is not minimal there exists $X \in \mathcal{B}_L$ such that $\mid \mathcal{S}_T \cap \mathcal{B}_L \mid \geq 2$ for all elements $\mathcal{S}_T$ of $\mathcal{L}$ containing X. Denote by $\mathcal{S}_{T_i}$, $i = 1, \ldots, q^t + 1$, the lines of Π through X. Let Y_i be an element of $\mathcal{S}_{T_i} \cap \mathcal{B}_L$ different from X, and let P be a point of $X \cap L$ and P_i a point of $Y_i \cap L$. The lines $\langle P, P_i\rangle$ are contained in the subspace L and they are mutually distinct because $T_i \cap T_j = X$ for $i \neq j$. Hence L contains $q^t + 1$ lines through P and this is impossible. So $\mathcal{B}_L$ is minimal.

If there exists an $\mathcal{S}_T \in \mathcal{L}$ such that $L \subset T$ we have immediately that $\mathcal{B}_L = \mathcal{S}_T$. If $\mathcal{B}_L = \mathcal{S}_T$, then for every $X \in \mathcal{S}_T$ we have $X \cap L \neq \emptyset$, i. e. $|T \cap L| \geq q^t + 1$ and this implies $L \subset T$. Moreover

$$\mid \mathcal{B}_L \mid \leq \sum_{X \in \mathcal{B}_L} \mid X \cap L \mid = \mid L \mid = q^t + \cdots + q + 1. \qquad \blacksquare$$

Corollary 1. *All linear blocking sets are small and minimal.*

If there exists $\mathcal{S}_T \in \mathcal{L}$ such that $\dim(T \cap L) = t-1$ then $\mathcal{B}_L$ is of Rédei type with respect to the line $\mathcal{S}_T$. So in this class of examples there are small minimal blocking sets of Rédei type. Using the classification of small blocking sets of Rédei type in [1] G. Lunardon has shown the following result:

Theorem 1. ([6], Thm.10) *Every small minimal blocking set of Rédei type, with either $p^e > 3$ or $p^e = 3$ and $|B| = q + q/3 + 1$, is linear.*

The following theorems give some condition in order to obtain linear blocking sets not of Rédei type.

Theorem 2. *Let $\mathcal{S}$ be a normal $(t-1)$-spread and let L be a t-dimensional subspace of $PG(3t-1,q)$, with $t \geq 4$. Suppose there is an element $\mathcal{S}_T$ of $\mathcal{L}$ such that:*

1) *$L \cap T$ is a $(t-2)$-dimensional subspace;*
2) *for all $X \in \mathcal{S}_T$, $X \cap L$ is either empty or a point.*

Then $\mathcal{B}_L$ is a small blocking set of $\Pi \cong PG(2,q^t)$ of size either $q^t + \cdots + q^2 + 1$ or $q^t + \cdots + q^2 + q + 1$. Moreover $\mathcal{B}_L$ is not of Rédei type.

Proof. Let X be a fixed element of $\mathcal{S}_T$. If $\mathcal{S}_{T'}$ is a line of Π incident with X and different from $\mathcal{S}_T$, we have

$$\dim \left\langle T' \cap L, T \right\rangle = \dim(T' \cap L) + \dim T - \dim(L \cap X) = \dim(T' \cap L) + 2t - 1 - \varepsilon,$$

where $\varepsilon \in \{-1, 0\}$, because, by hypothesis, $X \cap L$ is either empty or a point.

Hence, since $\left\langle T' \cap L, T \right\rangle \subset \langle T, L \rangle$ and $\dim \langle T, L \rangle = 2t + 1$ we obtain

$$\dim(T' \cap L) \leq \varepsilon + 2. \tag{1}$$

If Y_1 and Y_2 are two distinct elements of $\mathcal{S} \setminus \mathcal{S}_T$, from (1) it follows that $\dim(\langle Y_1, Y_2 \rangle \cap L) \leq 2$, so at most one of the subspaces $Y_1 \cap L$ and $Y_2 \cap L$ can be a line. Moreover if $\mathcal{S}_{T'}$ is a line of Π different from $\mathcal{S}_T$, $|\mathcal{S}_{T'} \cap \mathcal{B}_L|$ is either 1, or $q+1$, or q^2+1, or q^2+q+1, because by (1) $T' \cap L$ is a point or a line or a plane.

So, as $|\mathcal{S}_T \cap \mathcal{B}_L| = q^{t-2} + \cdots + q + 1$ and $t \geq 4$, if $\mathcal{B}_L$ is a Rédei blocking set, $\mathcal{S}_T$ must be a Rédei line.

Suppose that $\mathcal{B}_L$ is of Rédei type, and let X be an element of $\mathcal{S}$ contained in T and disjoint from L; so the lines through X distinct by $\mathcal{S}_T$ are all tangent lines to $\mathcal{B}_L$, and, as at most one $Y \in \mathcal{S} \setminus \mathcal{S}_T$ can contain a line of L, and $|L \setminus (L \cap T)| = q^t + q^{t-1}$, we have

$$q^t + q^{t-1} \leq q + 1 + q^t - 1 \;\Rightarrow\; q^{t-1} \leq q \;\Rightarrow\; t \leq 2,$$

and this is a contradiction, hence $\mathcal{B}_L$ is not of Rédei type.

Now considering the lines $\mathcal{S}_{T'}$ through X as above we obtain that if there is an $Y \in \mathcal{S} \setminus \mathcal{S}_T$ such that $\dim(Y \cap L) = 1$, as the elements of $\mathcal{S}$ different from Y contain at most one point of L we have

$$|\mathcal{B}_L| = q^{t-2} + \cdots + q^2 + q + 1 + q^t + q^{t-1} - q - 1 + 1 = q^t + \cdots + q^2 + 1,$$

otherwise

$$|\mathcal{B}_L| = q^t + \cdots + q^2 + q + 1,$$

and the proof is complete. ∎

Theorem 3. *Let $\mathcal{S}$ be a normal $(t-1)$-spread and let L be a t-dimensional subspace of $PG(3t-1,q)$, with $t \geq 4$. Suppose there is an element $\mathcal{S}_T$ of $\mathcal{L}$ such that:*

1) *$L \cap T$ is a $(t-r)$-dimensional subspace, with $2 \leq r \leq t/2$;*
2) *for all $X \in \mathcal{S}_T$, $X \cap L$ is either empty or a point;*
3) *there is $Y \in \mathcal{S} \setminus \mathcal{S}_T$ such that $Y \cap L$ is a $(r-1)$-dimensional subspace.*

Then $\mathcal{B}_L$ is a small blocking set of $\Pi \cong PG(2,q^t)$ of size $q^t + q^{t-1} + \cdots + q^r + 1$. Moreover $\mathcal{B}_L$ is not of Rédei type.

The proof is similar to the proof of Theorem 2.

3. The constructions

In this section we construct examples of small minimal blocking sets that are not of Rédei type.

Let V be a 3-dimensional vector space over the field $GF(q^t)$, and let $PG(2,q^t) = PG(V)$. We can regard V as a vector space of dimension $3t$ over $GF(q)$, in such case we use the symbol V'. Each point P of $PG(2,q^t)$ defines a $(t-1)$-subspace X_P of the projective space $PG(V') \cong PG(3t-1,q)$, and each line $l = \langle P,Q \rangle$ of $PG(2,q^t)$ defines a $(2t-1)$-subspace $T = \big\langle X_P, X_Q \big\rangle$.

The set $\mathcal{S} = \{X_P : P \in PG(V)\}$ is a normal $(t-1)$-spread of $PG(3t-1,q)$, so $\Pi = (\mathcal{S},\mathcal{L})$ is isomorphic to $PG(2,q^t)$ (see [5], [8]).

Denote by (X_0,X_1,X_2) the projective coordinates of a point of $PG(2,q^t)$, let l_∞ be the line with equation $X_2 = 0$ and let T_∞ be the corresponding $(2t-1)$-dimensional subspace of $PG(3t-1,q)$. The set $M = \{(a,a^q,0) : a \in GF(q^t) \setminus \{0\}\}$ of points of l_∞ is $GF(q)$-linear, hence it can be regarded as a $(t-1)$-dimensional subspace of $PG(3t-1,q)$, denoted by M', which is contained in T_∞. It is easy to see that for every $X \in \mathcal{S} \setminus \mathcal{S}_{T_\infty}$ we have $|X \cap M'| \in \{0,1\}$.

From now on we suppose $t \geq 4$.

Example 1. Let U be a $(t-2)$-dimensional subspace of M' and let A and B be two distinct points such that the line $\langle A,B \rangle$ is disjoint from T_∞. We notice that it is always possible to construct the line $\langle A,B \rangle$ disjoint from T_∞. The t-dimensional subspace $L = \langle U,A,B \rangle$ satisfies the conditions of Theorem 2, hence $\mathcal{B}_L$ is a linear blocking set which is not of Rédei type.

Example 2. Let U be a $(t-r)$-dimensional subspace of M', with $2 \leq r \leq t/2$, and let W be a $(r-1)$-dimensional subspace of $PG(3t-1,q)$ such that there exists an element of $\mathcal{S} \setminus \mathcal{S}_{T_\infty}$ containing W. The t-dimensional subspace $L = \langle U,W \rangle$ satisfies the conditions of Theorem 3, hence $\mathcal{B}_L$ is a linear blocking set which is not of Rédei type.

Hence we have proved the following theorem:

Theorem 4. *There exists a small minimal blocking set not of Rédei type in every projective plane $PG(2,q^t)$, with q a prime power and $t \geq 4$.*

References

[1] S. Ball, A. Blokhuis, A.E. Brouwer, L. Storme and T. Szőnyi: On the number of slopes of the graph of a function defined on a finite field, to appear in *Journal of Combinatorial Theory A*.

[2] A. Blokhuis: Blocking sets in Desarguesian Planes, in "Combinatorics: Paul Erdős is Eighty", **2**, D. Miklós, V.T. Sós, T. Szőnyi editors, Bolyai Soc. Math. Studies, (1996), 133–155.

[3] A. Blokhuis: Special pointsets in finite planes, *CWI Quarterly*, **9** (1997), 31–36.

[4] A. E. Brouwer and H. A. Wilbrink: Blocking sets in translation planes, *J. Geom*, **19** (1982), 200.

[5] G. Lunardon: Normal spreads, submitted.

[6] O. Polverino: Small blocking sets in $PG(2,p^3)$, submitted to *Designs, Codes and Cryptography*.

[7] B. Segre: Teoria di Galois, fibrazioni proiettive e geometrie non desarguesiane, *Ann. Mat. Pura Appl.*, **64** (1964), 1–76.

[8] T. Szőnyi: Blocking sets in desarguesian affine and projective planes, *Finite Fields and Appl.*, **3** (1997), 187–202.

Pompeo Polito

Dip. di Matematica e Applicazioni
Complesso di Monte S. Angelo - Edificio T
Via Cintia
80126 Napoli, Italy
polito@matna2.dma.unina.it

Olga Polverino

Dip. di Matematica e Applicazioni
Complesso di Monte S. Angelo - Edificio T
Via Cintia
80126 Napoli, Italy
polverino@matna2.dma.unina.it